Lecture 2: Introduction and Empirical framework for Markov perfect industry dynamics

April 15, 2015
Introduction:

- Importance of strategic interactions: games vs single-agent problems
  - In single-agent problems you search for a solution to a dynamic programming problem, i.e., value function is a fixed point of the Bellman equation.
  - In multi-agent models an equilibrium is a fixed point of a system of best response operators and each player’s best response is itself a solution to a dynamic programming problem.

- Several problems are specific to a game-theoretic environment
  - Existence of an equilibrium.
  - Multiplicity and selection of equilibria.
  - Computational burden.
Introduction:

• We seek for a framework that is consistent with several empirical facts about imperfectly competitive industries
  ○ Simultaneous exit and entry.
  ○ Heterogeneity among firms within a market.
  ○ Firm- and market-level uncertainty.
  ○ Correlations in the outcomes (profits) among firms in the same industry.

• We begin our discussion with a widely used framework: Ericson and Pakes (1995).

• EP framework is a dynamic stochastic game with a discrete state space.
Aside: why this is useful

• There are several methods that allow us to recover underlying primitives of dynamic games without ever solving for an equilibrium, e.g.,
  ○ Bajari, Benkard, and Levin (2007),
  ○ Pakes, Ostrovsky, and Berry (2007),
  ○ Aguirregabiria and Mira (2007),
  ○ Pesendorfer and Schmidt-Dengler (2008), etc.

• Why do we bother to go through the hassle of computing one?
  ○ Academic curiosity
  ○ Counterfactual simulations

• Even if one never uses the model to estimate parameters of interest using brute force approach (e.g., nested fixed point), computing the Markov perfect equilibrium is necessary to evaluate various policies.
EP framework: setup

- A modified version of the Ericson-Pakes framework based on Doraszelski and Pakes (2007)

- Forward-looking firms, discrete time, infinite horizon.

- Firms maximize expected discounted sum of profit flows.

- At the beginning of each period, two sets of firms simultaneously make decisions:
  1. Incumbent firms decide whether to stay in or not. Conditional on staying - how much to invest.
  2. Potential entrants decide whether to enter the market, and if they do, how much to invest.

- Realizations of these decisions take place at the end of the time period.
EP framework: setup

- Product market competition takes place
  - Outcomes in the product market do not affect the dynamics of the industry
    
    This is called “static-dynamic breakdown”. Does this always hold?
  - Treat per-period profit function as a primitive of the dynamic game

- Per-period profits realize (depending on the state and actions)

- Timing:
  - Beginning of $t$: (1) investment, (2) entry, and (3) exit decisions are made simultaneously
  - End of $t$: realizations of the decisions made and product market competition takes place (note: prior to exiting an incumbent collects current period profit).
EP framework: setup

- **Incumbent firms**
  - Firm $i$ is characterized by its state $\omega_i \in \Omega = \{1, 2, 3, \ldots, \bar{\omega}\}$
  - Investment $x_i \geq 0$ may increase its state: $\omega_i$ in the next period is stochastically increasing in $x_i$.
    
    At the same time, $\omega_i$ may decrease by an exogenous negative shock.
  - At the beginning of each period, incumbent firm $i$ draws a random and privately known scrap value $\phi_i$ from $F$.
    
    We assume that $\phi_i$ is iid across firms and time. Why is this helpful?
  - If $i$ decides to exit, it competes in the product market in the last period and then exits.
  - We use $r_i$ to denote the probability that the incumbent firm $i$ stays in the industry.
EP framework: setup

- **Potential Entrants**
  - Finite number $\mathcal{E}$ of potential entrants in each time period.
  - Potential entrants are short-lived, i.e. no strategic delays in entry decisions.
  - At the beginning of each period, each potential entrant draws a random and privately known entry cost $\phi_i^e$ from $F^e$.
    
    We assume $\phi_i^e$ are iid across firms and time.
  - Upon entry, potential entrant $i$ incurs $\phi_i^e$ and chooses its initial investment $x_i^e \geq 0$.
  - It earns profits from the next period.
  - We use $r_i^e$ to denote probability that potential entrant $i$ enters.
EP framework: setup

- **States**
  - A list of states of the incumbent firms $\omega = (\omega_1, \ldots, \omega_n)$
  - A vector of scrap values $\phi = (\phi_1, \ldots, \phi_n)$, and
  - A vector of entry costs $\phi^e = (\phi^e_1, \ldots, \phi^e_n)$

  completely characterize the industry

- To make computations easier, integrate out $\phi$ and $\phi^e$: now $\omega$ completely characterizes the industry.

- Industry structure is given by

  $$S = \{(\omega_1, \ldots, \omega_n) : \omega_i \in \Omega, n \leq \bar{n}\}$$
EP framework: Payoffs and strategies

- Payoffs and strategies are symmetric and anonymous
  - Symmetry: $f_i(\omega_i, \omega_{-i}) = f_j(\omega_i, \omega_{-i})$
  - Anonymity (exchangeable): $f(\omega_i, \omega_{-i}) = f(\omega_i, \omega_{\pi(-i)})$
  - Asymmetries between firms are then captured by the state variables only.

- This allows us to focus on a proper subset $S^o$ of $S$:

$$S^o = \left\{ (\omega_i, s) : \omega_i \in \Omega, s = (s_1, \ldots, s_{\bar{\omega}}), s_{\bar{\omega}} \in \mathbb{Z}^+, \sum_{\omega \in \Omega} s_{\omega} \leq \bar{n} \right\}$$

where $\mathbb{Z}^+$ is the set of non-negative integers.

- Let $\pi(\omega_i, \omega_{-i})$ denote the per-period profit in the product market when the industry’s state is $\omega$. 
EP framework: Transitions

• Transition probability of firm \( i \)'s state depends only on its own investment and current state (not on the investments or states of its competitors).

• Let \( \eta \) denote industry-wide shock, e.g. change in demand, input costs, technology, etc.

   We need this to allow for positive correlation in profits of different firms in the same industry.

• A family of probability distributions for \( \omega'_i \) given current state \( \omega_i \), firms' choices, \( x_i \), and industry shocks \( \eta \):

\[
\mathcal{P}_{\omega'} \equiv \left\{ p(\cdot | \omega_i, x_i, \eta) : \omega_i \in \Omega, x_i \in \mathbb{R}^+, \eta \in \Upsilon \right\},
\]

where we assume \( \mathcal{P}_{\omega'} \) is stochastically increasing in \( \omega_i \) and \( x_i \).
EP framework: Transitions

- Example of the evolution of $\omega$

$$\omega_i' = \omega_i + \nu_i - \eta,$$

where $\nu_i$ represents an outcome of the firm's investment.

- Example of the investment outcome distribution

$$\Pr(\nu = 1|x_i) = \frac{\alpha x_i}{1 + \alpha x_i}, \quad \nu \in \{0, 1\}$$
EP framework: Transitions
EP framework: Transitions

• Consider a game of capacity accumulation with $n = 2$, $\bar{\omega} \geq 3$ states, and no entry/exit.

• In each period, firms invest and the likelihood of positive outcome is given by $p_i = \Pr(\nu = 1|x_i)$.

• An independent across firms depreciation shock hits a firm with probability $\delta$ and if it does, it decreases the firm’s capacity by one.

• Let $\bar{\delta} = 1 - \delta$ then transition probabilities are given in the following table.
EP framework: Transitions

- Example from Doraszelski and Satterthwaite (2010)

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EP framework: Incumbent’s problem

- Let $V(\omega_i, \omega_{-i}, \phi)$ denote the expected NPV of all future cash flows of firm $i$, when the industry is characterized by $\omega$ and the firm has drawn $\phi$. The Bellman equation is

$$V(\omega_i, \omega_{-i}, \phi) =$$

$$\pi(\omega_i, \omega_{-i}) + \max\left\{\phi, \max_{x_i} \left( -x_i + \beta E \left[ V(\omega'_i, \omega'_{-i}, \phi') | \omega_i, \omega_{-i}, x_i \right] \right) \right\}$$

- Let $q(\omega'_{-i} | \omega_i, \omega_{-i}, \eta)$ denote perceived probability of the next period value of its competitors’ state $\omega'_{-i}$. Then,

$$E \left[ V(\omega'_i, \omega'_{-i}, \phi') | \omega_i, \omega_{-i}, \phi \right] = \sum_{\nu} W(\nu | \omega_i, \omega_{-i}) p(\nu | x_i),$$

$$W(\nu | \omega_i, \omega_{-i}) \equiv \sum_{\omega'_{-i}, \eta} \int_{\phi'} V(\omega_i + \nu - \eta, \omega'_{-i}, \phi') dF(\phi') q(\omega'_{-i} | \omega_i, \omega_{-i}) p(\eta)$$
Example for $n = 2$, $E = 1$, $\omega = (1, 2, 0)$ and $\eta = 0$.

For given policies $r(\omega_i, \omega_{-i})$, $r^e(\omega)$, $x_i(\omega_i, \omega_{-i})$, $i = 1, 2$, and $x^e(\omega)$, $q(\omega' - i|\omega_i, \omega_{-i}, \eta)$ is calculated as

$$q(\omega'_{-1} = (2, 1)|\omega, \eta) = r(\omega_2, \omega_{-2})(1 - \Pr(\nu = 1|x_2(\omega_2, \omega_{-2})))$$

$$\times r^e(\omega) \Pr(\nu = 1|x^e(\omega))$$
EP framework: Incumbent’s problem

• For given \( \{ W(\cdot) \} \), the optimal investment behavior is characterized by

\[
x_i \left( \beta \sum_{\nu} W(\nu|\omega_i, \omega_{-i}) \frac{\partial p(\nu|x_i)}{\partial x_i} - 1 \right) = 0 \land x_i \geq 0
\]

○ Firm \( i \) selects an investment level equalizing current marginal costs with the marginal change in the expected present value of the states that might be realized next period.

• Let \( x(\omega_i, \omega_{-i}) \) be an optimal investment policy and \( \chi(\omega_i, \omega_{-i}, \phi) \) be an indicator function which equals to one if the firm continues and zero otherwise.
EP framework: Incumbent’s problem

- Then

\[
\chi(\omega_i, \omega_{-i}, \phi) = \arg \max_{\chi \in \{0, 1\}} (1 - \chi) \phi \\
+ \chi \left( \beta \sum_{\nu} W(\nu|\omega_i, \omega_{-i}) p(\nu|x(\omega_i, \omega_{-i})) - x(\omega_i, \omega_{-i}) \right)
\]

and

\[
r(\omega_i, \omega_{-i}) = F \left( \beta \sum_{\nu} W(\nu|\omega_i, \omega_{-i}) p(\nu|x(\omega_i, \omega_{-i})) - x(\omega_i, \omega_{-i}) \right)
\]

defines probability of staying in the industry.
EP framework: Incumbent’s problem

- Using this, integrate $\phi$ to get

$$V(\omega_i, \omega_{-i}) = \pi(\omega_i, \omega_{-i}) + (1 - r(\omega_i, \omega_{-i}))E[\phi|\chi(\omega_i, \omega_{-i}, \phi) = 0]$$

$$+ r(\omega_i, \omega_{-i}) \left( \beta \sum_{\nu} W(\nu|\omega_i, \omega_{-i}) p(\nu|x(\omega_i, \omega_{-i})) - x(\omega_i, \omega_{-i}) \right)$$

- where

$$E[\phi|\chi(\omega_i, \omega_{-i}, \phi) = 0] = \frac{1}{1 - r(\omega_i, \omega_{-i})} \int_{\phi > F^{-1}(r(\omega_i, \omega_{-i}))} \phi dF.$$
EP framework: Entrant’s problem

- Value function for a potential entrant $i$ is given by

$$V^e(\omega, \phi^e) = \max \left\{ 0, \max_{x^e_i} \left[ -\phi^e - x^e_i + \beta \sum_{\nu} W^e(\nu|\omega)p(\nu|x^e_i) \right] \right\}$$

where $W^e(\cdot)$ is defined similar to $W(\cdot)$.

- Optimal investment policy is characterized by

$$x^e_i \left( \beta \sum_{\nu} W^e(\nu|\omega) \frac{\partial p(\nu|x^e_i)}{\partial x^e_i} - 1 \right) = 0 \land x^e_i \geq 0$$

- Let $x^e(\omega)$ be $i$’s optimal investment policy and $\chi^e(\omega, \phi)$ be the indicator function which equals to one if the potential entrant enters and zero otherwise.
EP framework: Entrant’s problem

- Then,

$$\chi^e(\omega, \phi^e) = \arg \max_{\chi \in \{0,1\}} \chi \left( -\phi^e - x^e(\omega) + \beta \sum_{\nu} W^e(\nu|\omega)p(\nu|x^e(\omega)) \right)$$

and

$$r^e(\omega) = F^e \left( -x^e(\omega) + \beta \sum_{\nu} W^e(\nu|\omega)p(\nu|x^e(\omega)) \right)$$

defines entry probability.
EP framework: Entrant’s problem

- Using this, integrate $\phi$ to get

$$V^e(\omega) = r^e(\omega) \left[ -E[\phi^e|\chi^e(\omega, \phi^e) = 1] - x^e(\omega) + \beta \sum_{\nu} W^e(\nu|\omega)p(\nu|x^e(\omega)) \right],$$

where

$$E[\phi^e|\chi^e(\omega, \phi^e) = 1] = E\left[ \phi^e|\phi^e < F^{-1}(r^e(\omega)) \right]$$

$$= \frac{1}{r^e(\omega)} \int_{\phi^e < F^{-1}(r^e(\omega))} \phi^e dF^e$$
EP framework: Equilibrium

- We consider a Markov perfect equilibria (MPE)

- At each $\omega \in S^o$ each incumbent and each potential entrant
  - chooses optimal policies given its beliefs about future industry structures,
  - these beliefs are consistent with the behavior of each agent’s competitors

- More formally: MPE consists of $\{V(\cdot), V^e(\cdot), \chi(\cdot), \chi^e(\cdot), x(\cdot), x^e(\cdot)\}$ at each $\omega \in S^o$, such that
  - given policies $\{\chi(\cdot), \chi^e(\cdot), x(\cdot), x^e(\cdot)\}$, $V(\cdot)$ solves the Bellman equation for incumbent firms, and $V^e(\cdot)$ solves the Bellman equation for entrants $\forall \omega \in S^o$.
  - given value functions $V$ and $V^e$, the policies for both incumbents and potential entrants satisfy the optimality conditions.
EP framework: Equilibrium existence

A1. Boundedness of primitives:
   (i) finite state space, $\bar{\omega} < \infty$, $\bar{n} < \infty$;
   (ii) bounded profits, $\bar{\pi} < \infty$ s.t. $-\bar{\pi} < \pi(\omega) < \bar{\pi}$, $\forall \omega$;
   (iii) bounded investments, $\bar{x} < \infty$ and $\bar{x}^e < \infty$;
   (iv) distributions $F$ and $F^e$ have positive densities over connected supports with $\bar{\phi} < \infty$ and $\bar{\phi}^e < \infty$, s.t.
   \[ -\bar{\phi} < \int |\phi| dF(\phi) < \bar{\phi} \quad \text{and} \quad -\bar{\phi}^e < \int |\phi^e| dF(\phi^e) < \bar{\phi}^e \]
   (v) discounting, $\beta \in [0, 1)$.

A2. $P(\omega', \omega, \chi(\omega_i, \omega_{-i}, \phi), \chi^e(\omega, \phi), x(\omega_i, \omega_{-i}), x^e(\omega))$ is a continuous function of $x(\omega_i, \omega_{-i})$ and $x^e(\omega)$ for all $\omega', \omega$, and all $\chi(\omega_i, \omega_{-i}, \phi)$ and $\chi^e(\omega, \phi)$.

A3. There exists a unique $x(\omega_i, \omega_{-i})$ that attains maximum of $V(\omega_i, \omega_{-i}, \phi)$ for all $V$, $r_{-i}$, $r_{-i}^e$, $x_{-i}$, $x_{-i}^e$, and $\omega$. There exists a unique $x^e(\omega)$ that attains max of $V^e(\omega, phi^e)$ for all $V$, $r_{-i}$, $r_{-i}^e$, $x_{-i}$, $x_{-i}^e$, and $\omega$. 
EP framework: Equilibrium existence

- **Proposition 2.** Under Assumptions 1, 2, and 3, an equilibrium exists in cutoff entry/exit and pure investment strategies. (Doraszelski and Satterthwaite (2010))
  - proof relies on a continuous mapping from policies into themselves (Brouwer’s fixed-point theorem);
  - adding random scrap values and entry costs allows us to treat the continuous exit and entry probabilities as policies;
  - given strategies of competitors, a firm solves a single-agent decision problem, can use dynamic programming techniques (contraction mapping) to show that the firm’s best reply is well-defined.
EP framework: Equilibrium existence

- Example of *non-existence* with deterministic scrap values from Doraszelski and Satterthwaite (2010)
  - Consider a war of attrition with a common and constant exit value $\phi$.
  - No entry, state $\omega = (\omega_1, \omega_2)$, where $\omega_i = 1$ denotes active and $\omega_i = 2$ inactive state.
  - Profits:
    $$\pi(\omega_i, \omega_{-i}) = \begin{cases} 
    \pi(1, 1) & \text{duopoly} \\
    \pi(1, 2) & \text{monopoly} 
    \end{cases}$$
  - Let $\phi$ be s.t.
    $$\frac{\beta \pi(1, 1)}{1 - \beta} < \phi < \frac{\beta \pi(1, 2)}{1 - \beta}.$$
EP framework: Equilibrium existence

- Given firm 2’s decision \( \chi(1, 1) \in \{0, 1\} \), firm 1’s values are

\[
V(1, 2) = \sup_{\tilde{\chi}(1, 2) \in \{0, 1\}} \pi(1, 2) + (1 - \tilde{\chi}(1, 2))\phi + \tilde{\chi}(1, 2)\beta V(1, 2)
\]

\[
V(1, 1) = \sup_{\tilde{\chi}(1, 1) \in \{0, 1\}} \pi(1, 1) + (1 - \tilde{\chi}(1, 1))\phi
\]

\[
+ \tilde{\chi}(1, 1)\beta \{\chi(1, 1)V(1, 1) + (1 - \chi(1, 1))V(1, 2)\}
\]

- Firm 1’s optimal exit decision \( \tilde{\chi}(1, 2) \) and \( \tilde{\chi}(1, 1) \) satisfy

\[
\tilde{\chi}(1, 2) = 1\{\phi \leq \beta V(1, 2)\}
\]

\[
\tilde{\chi}(1, 1) = 1\{\phi \leq \beta[\chi(1, 1)V(1, 1) + (1 - \chi(1, 1))V(1, 2)]\}
\]

- Consider a symmetric equilibrium in pure exit strategies.
EP framework: Equilibrium existence

- Suppose $\chi(1, 2) = 0$. Then $V(1, 2) = \pi(1, 2) + \phi$ and (1) implies

  $$\phi \geq \beta(\pi(1, 2) + \phi) \Rightarrow \phi \geq \frac{\beta \pi(1, 2)}{1 - \beta},$$

  contradiction.

- Suppose $\chi(1, 1) = 1$. Then $V(1, 1) = \frac{\pi(1, 1)}{1 - \beta}$ and (2) implies

  $$\phi \leq \frac{\beta \pi(1, 1)}{1 - \beta},$$

  contradiction.

- Finally, $\chi(1, 2) = 1$ and $\chi(1, 1) = 0$. Then $V(1, 2) = \frac{\pi(1, 2)}{1 - \beta}$ and (2) implies

  $$\phi \geq \beta V(1, 2) \Rightarrow \phi \geq \frac{\beta \pi(1, 2)}{1 - \beta},$$

  contradiction.
EP framework: Equilibrium existence

- To ensure existence consider random scrap values and entry costs
- Setup is the same except that scrap values now are given by $\phi + \epsilon \theta_i$, where $\phi$ is a constant and $\theta_i \sim F(\cdot)$ with $E[\theta] = 0$.
- Private information is $\theta$ and $\epsilon$ is a constant that measures the importance of incomplete information.
- Bellman equation

$$V(1, 2) = \sup_{\tilde{r}(1, 2) \in [0, 1]} \pi(1, 2) + (1 - \tilde{r}(1, 2)) \phi$$

$$+ \epsilon \int_{\theta > F^{-1}(\tilde{r}(1, 2))} \theta dF(\theta) + \tilde{r}(1, 2) \beta V(1, 2)$$

$$V(1, 1) = \sup_{\tilde{r}(1, 1) \in [0, 1]} \pi(1, 1) + (1 - \tilde{r}(1, 1)) \phi$$

$$+ \tilde{r}(1, 1) \beta [r(1, 1)V(1, 1) + (1 - r(1, 1))V(1, 2)]$$
EP framework: Equilibrium existence

- Optimal exit decisions of firm 1 are given by

\[
\tilde{r}(1, 2) = F \left( \frac{\beta V(1, 2) - \phi}{\epsilon} \right)
\]
\[
\tilde{r}(1, 1) = F \left( \frac{\beta [r(1, 1)V(1, 1) + (1 - r(1, 1))]V(1, 2)] - \phi}{\epsilon} \right)
\]

- In a symmetric equilibrium, \( \tilde{r}(\omega_1, \omega_2) = r(\omega_2, \omega_1) \), yielding a system of four equations with four unknowns: \( V(1, 2) \), \( V(1, 1) \), \( r(1, 2) \), and \( r(1, 1) \)

- Note that equilibrium with random scrap values converges to the equilibrium in mixed strategies as \( \epsilon \) approaches zero.
EP framework: Equilibrium uniqueness

- In general, multiple equilibria cannot be excluded.

- Three reasons (see examples in the online Appendix to Doraszelski and Satterthwaite (2010))
  - investment decisions,
  - entry/exit decisions,
  - product market competition.

- How to deal with them?
  1. Assume away.
  2. Change endogenous variables so there exists a unique equilibrium in terms of the new endogenous variable in question.
EP framework: Important properties

- Equilibrium is characterized by substantial inflows and outflows of firms
- Multiple rank reversals are possible during the life of any collection of firms
- All firms die in finite time almost surely, but new firms continually enter
- Ericson and Pakes (1995) prove that the state space $S$, contains a unique, positive recurrent communicating class $R \subset S$,
  - Define $\mu_n(v)$ as the probability that the structure of the industry with initial distribution $v$ is in state $s$ after $n$ periods, then there exists a unique, invariant probability measure, $\mu^*$, on $S$ s.t.
    \[ \forall s \in S, \mu_n(s) \underset{n \to \infty}{\longrightarrow} \mu^* \]
    \[ \mu_s^* = 0 \forall s \in S \setminus R \]
    and the distribution of a stationary ergodic Markov process, $P_{\mu^*}$, with transition $Q$ s.t. $\mu^* Q = \mu^*$. 
EP framework: Important properties

- Implications of ergodicity (in our context we can say that time averages approximate state averages)
  - All industry structures in the recurrent class $R \subset S$ are realized infinitely often. $R$ is often much smaller than $S$. Note that the industry evolves in a non-degenerate, but increasingly regular way, i.e. there is never a “limit” structure of the industry.
  - After some time, a certain stochastic regularity will appear in the evolution of the industry.
  - Effects of any initial situation systematically fade. Two possible histories for the industry with different initial conditions, once they intersect in any state, have identical distributions over future sample paths conditional on that intersection.


